Some remarks on the complex heat kernel on $\mathbb{C}^\nu$ in the scalar potential case*†

Thierry Hargé

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Abstract

In previous works, we used a so-called deformation formula in order to study, in particular, the Borel summability of the heat kernel of some operators. A goal of this paper is to collect miscellaneous remarks related to these works. Here the complex setting plays an important role. Moreover, the deformation formula provides a solution of the heat equation in "unusual" cases. We also give a uniqueness statement concerning these cases.

1 Introduction

In previous works [Ha4, Ha5], we used a so-called deformation formula in order to study the Borel summability of the heat kernel, $p(t,x,y)$, associated to a ν-dimensional partial differential operator ($\nu \in \mathbb{N}^*$). This formula is extended to the non-autonomous case [Ha6]. The natural setting for this formula is a complex one ($t \in \mathbb{C}, \mathcal{R}et \geqslant 0$ and $x, y \in \mathbb{C}^\nu$). For instance this formula is valid for operators as

$$P := \partial_x^2 + \lambda x^2 + c(x) := \partial_{x_1}^2 + \cdots + \partial_{x_\nu}^2 + \lambda(x_1^2 + \cdots + x_\nu^2) + c(x_1, \ldots, x_\nu) \quad (1.1)$$

where $\lambda \in \mathbb{R}$ and the function c is the Fourier transform of a suitable Borel measure. The aim of this paper is threefold:

1. Defining in a unique way the heat kernel associated to the operator P is a well known procedure if $\lambda \leqslant 0$. If $\lambda > 0$, one can use the commutator theorem [Re-Si] for instance (see Remark 3.5). However, we look for a statement adapted to a full complex setting and covering the non-autonomous case: $P = P_0 + c$ where P_0 is defined by (3.1). Here is the purpose of Proposition 3.2. The statement and the proof of this proposition are standard. We assume that

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the coefficients defining P_0 satisfy a reality-preserving property; see (3.2). This implies that the operator $P_0|_{i\mathbb{R}}$ is symmetric with respect to the L^2 inner product. The unicity is therefore a consequence of the conservation of the L^2-norm for the solutions of the time dependent Schrödinger equation associated to P_0. Let us make the following remarks.

- In our setting, the Schrödinger kernel is viewed as the boundary value of the heat kernel for imaginary[1] values of t.
- Our goal is not to solve the heat equation but to study the heat kernel as a function defined on some subset of $\mathbb{C}_{\sqrt{\cdot}} \times \mathbb{C}^{2\nu}$ ($\mathbb{C}_{\sqrt{\cdot}}$ denotes the Riemann[2] surface of the square root function).

About the existence problem, we use [Ha4] and [Ha6].

2. Considering a complex setting allows one to make some remarks.

   We reformulate Proposition 3.2 by using the analytic dilation given by

$$(t, x, y) \mapsto (e^{i\epsilon}t, e^{i\epsilon/2}x, e^{i\epsilon/2}y) \ , \ \ \epsilon \in \mathbb{R}/4\pi\mathbb{Z} \tag{1.2}$$

   (see Corollary 4.1). As a consequence, we see that the deformation formula provides a solution for the Schrödinger equation

$$i^{-1}\partial_{\boldsymbol{t}} p = \big(\partial_x^2 + \lambda x^2 + c(x)\big)p \ , \ \ p|_{\boldsymbol{t}=0} = \delta_{x=y} \ , \ \ (\boldsymbol{t} \in \mathbb{R}, |\boldsymbol{t}| \ll 1, x, y \in \mathbb{R}^\nu)$$

   with fast growing potentials such as, for instance, $c(x) = e^{x^2}$ (see Corollary 4.3 and Remark 4.4). We do not state uniqueness in this case (this can be done by keeping the complex point of view for the space variables). This equation, in the free case ($\lambda = 0$), was considered by Kuna, Streit and Westerkamp [K-S-W]. In this paper, the authors build a Feynman integral dealing with such potentials. A version of the deformation formula can also be found there [K-S-W, Remark 19].

   Then we give a simple assumption on the potential c providing the existence of the heat kernel on a conical neighboorhood of $\mathbb{R}^+$ but with an aperture larger than $\pi/2$ (see Proposition 4.5).

3. We consider the Borel summability of the small time expansion of the conjugate of the heat kernel in the free case ($\lambda = 0$): we reformulate a statement given in [Ha4] by using the analytic dilation given by (1.2). As in Proposition 4.5, we consider two cases. We consider a simple class of potentials for which Borel-Nevalinna summability (see Section 5 for the definition) holds in an arbitrary direction. Then we give assumptions on the potential c implying Borel-Watson summability of this small time expansion instead of Borel-Nevalinna summability as in [Ha4]. Borel-Nevalinna summability uses

[1] With our notation, a physical interpretation of the parameter t is the reciprocal of the temperature which is a macroscopic variable. We denote by $\boldsymbol{t}$ the "physical" time.

[2] A ramification occurs in our statements if the dimension ν is odd: this is only due to the existence of the factor $t^{-\nu/2}$ in the expression of the free heat kernel.

that c is the Fourier transform of some Borel measure μ defined on $\mathbb{R}^\nu$ with a suitable convergence assumption. Borel-Watson summability uses that c is the Fourier transform of an analytic function defined on a conical neighbourhood of $\mathbb{R}^\nu$, with a similar convergence assumption. Let us remark that Borel-Watson summability of a series is a central tool when the critical time is a non-trivial power of the variable [Bals], [Ma-Ra].

We assume in this paper that the potential c is $\mathbb{C}$-valued. Our statements also hold if this potential is matrix-valued as in [Ha4] and [Ha6].

2 Notation

For $z \in \mathbb{C}$, we denote $\operatorname{sh} z := \frac{1}{2}(e^z - e^{-z})$, $\operatorname{ch} z := \frac{1}{2}(e^z + e^{-z})$. Let $\nu \geqslant 1$. For $\lambda, \mu \in \mathbb{C}^\nu$, we denote $\lambda \cdot \mu := \lambda_1\mu_1 + \cdots + \lambda_\nu\mu_\nu$, $\lambda^2 := \lambda \cdot \lambda$. These notations are extended to operators as $\partial_x := (\partial_{x_1}, \ldots, \partial_{x_\nu})$. We also denote $\bar\lambda := (\bar\lambda_1, \ldots, \bar\lambda_\nu)$, $|\lambda| := (\lambda \cdot \bar\lambda)^{1/2}$ (if $\lambda \in \mathbb{R}^\nu$, $|\lambda| = \sqrt{\lambda^2}$). We denote by $\boldsymbol{\pi}$ the canonical projection from $\mathbb{R}/4\pi\mathbb{Z}$ onto $\mathbb{R}/2\pi\mathbb{Z}$. For $\theta \in \mathbb{R}/4\pi\mathbb{Z}$, we denote by $e^{i\theta}$ the element of $\mathbb{C}_{\sqrt{\cdot}}$, the Riemann surface of the square root function, with argument θ and modulus 1. Then $\mathbb{C}_{\sqrt{\cdot}} := \{z = re^{i\theta} | r > 0, \theta \in \mathbb{R}/4\pi\mathbb{Z}\}$. For $z = re^{i\theta} \in \mathbb{C}_{\sqrt{\cdot}}$, we denote $z^{1/2} := r^{1/2}e^{i\theta/2}$. If $z \in \mathbb{C}$, we also denote by $z^{1/2}$ the square root of z which is defined up to a sign. Let $m \geqslant 1$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. For every subset A of $\mathbb{C}^m$, we denote $e^{i\theta}A := \{e^{i\theta}z | z \in A\}$. We also use this notation if $\theta \in \mathbb{R}/4\pi\mathbb{Z}$ and $A \subset \mathbb{C}_{\sqrt{\cdot}}$. If $\alpha \in \mathbb{R}/2k\pi\mathbb{Z}$ ($k \in \mathbb{N}^*$) and $r \in \mathbb{R}^+$, we denote by $]\alpha - r, \alpha + r[_{2k\pi}$ the open interval[3] of $\mathbb{R}/2k\pi\mathbb{Z}$ with end points $\alpha - r$ and $\alpha + r$. We denote

$$\mathbb{C}^+ := \{z \in \mathbb{C} | \mathcal{R}e(z) > 0\}\ ,\ \ \overline{\mathbb{C}^+} := \{z \in \mathbb{C} | \mathcal{R}e(z) \geqslant 0\}$$

and, if $T > 0$,

$$D_T := \{z \in \mathbb{C} | |z| < T\}\ ,\ \ D_T^+ := D_T \cap \mathbb{C}^+\ ,\ \ \bar D_T^+ := D_T \cap \overline{\mathbb{C}^+}.$$

Let $\mathfrak{B}$ denote the collection of all Borel sets on $\mathbb{R}^m$. An $\mathbb{C}$-valued measure μ on $\mathbb{R}^m$ is an $\mathbb{C}$-valued function on $\mathfrak{B}$ satisfying the classical countable additivity property (see [Ru]). We denote by $|\mu|$ the positive measure defined by

$$|\mu|(E) = \sup \sum_{j=1}^{\infty} |\mu(E_j)| (E \in \mathfrak{B}),$$

the supremum being taken over all partitions $\{E_j\}$ of E. In particular, $|\mu|(\mathbb{R}^m) < \infty$.

We denote by $\mathcal{D}(e^{i\theta}\mathbb{R}^m)$ the space of smooth functions with compact support defined on $e^{i\theta}\mathbb{R}^m$. If Ω is an open domain in $\mathbb{C}^m$, we denote by $\mathcal{A}(\Omega)$ the space of $\mathbb{C}$-valued analytic functions on Ω. These spaces are equipped with their

[3] $r > k\pi \Rightarrow]\alpha - r, \alpha + r[_{2k\pi} = \mathbb{R}/2k\pi\mathbb{Z}$.

standard Frechet structure (the semi-norms are indexed by compact sets and eventually differentiation order). If $U = e^{i\theta}]-T,T[$ or $e^{i\theta}\bar{D}_T^+$ or $\overline{\mathbb{C}^+}$ and $\mathcal{F}$ is a Frechet space, $\mathcal{C}^\infty\big(U,\mathcal{F}\big)$ denotes the standard Frechet space of smooth functions defined on U with values in $\mathcal{F}$. For instance, if the topology of $\mathcal{F}$ is defined by a family of semi-norms $(|\cdot|_j)_{j\in J}$, the Frechet structure of $\mathcal{C}^\infty\big(U,\mathcal{F}\big)$ is defined by the semi-norms $|\cdot|_{\alpha,j}$ ($|f|_{\alpha,j} = \sup_{t\in U}|\partial_t^\alpha f(t)|_j$) if $U = e^{i\theta}]-T,T[$ or $e^{i\theta}\bar{D}_T^+$. In the case $U = \overline{\mathbb{C}^+}$, suprema defining the semi-norms are taken over compact sets $\overline{\mathbb{C}^+}\cap D(0,R)$ where $R>0$. We now define global spaces (with respect to the space variable). We denote by $\mathcal{S}(\mathbb{R}^m)$ the space of Schwartz functions:

$$f\in\mathcal{S}(\mathbb{R}^m) \Leftrightarrow \forall(\alpha,\beta)\in\mathbb{N}^m\times\mathbb{N}^m,\ \sup_{z\in\mathbb{R}^m}|z^\alpha\partial_z^\beta f(z)|<\infty.$$

We consider the following spaces of smooth functions

- if $I =]-T,T[$ or $I = i]-T,T[$

$$f\in\mathcal{C}_{b,1}^\infty(I\times\mathbb{R}^m)\Leftrightarrow$$

$$\forall(\alpha,\beta)\in\mathbb{N}\times\mathbb{N}^m, \exists C>0, \forall(t,z)\in I\times\mathbb{R}^m, |\partial_t^\alpha\partial_z^\beta f(t,z)|\leqslant C(1+|z|)^\alpha.$$

- if $I=\mathbb{R}$ or $I = i\mathbb{R}$

$$f\in\mathcal{C}_b^\infty(I\times\mathbb{R}^m)\Leftrightarrow\forall(\alpha,\beta)\in\mathbb{N}\times\mathbb{N}^m,\ \sup_{(t,z)\in I\times\mathbb{R}^m}|\partial_t^\alpha\partial_z^\beta f(t,z)|<\infty.$$

3 The heat kernel viewed from the positive direction

3.1 The setting

Let U be a complex open neighbourhood of $0\in\mathbb{C}$. Let P_0 be the operator acting on $\mathcal{A}(U\times\mathbb{C}^\nu)$, defined by

$$P_0 = A(t)\cdot(\partial_x + B(t)x)\otimes(\partial_x+B(t)x) - C(t)\cdot x\otimes x \tag{3.1}$$

where A, B and C are $\nu\times\nu$ complex matrix-valued analytic functions on U. We assume that the matrices A and C are symmetric and that the matrix $A(0)$ is real positive definite. We assume that

$$\text{the functions } A|_{i\mathbb{R}}, (iB)|_{i\mathbb{R}} \text{ and } C|_{i\mathbb{R}} \text{ are real-valued near } 0. \tag{3.2}$$

Then the equation

$$\partial_t u = P_0 u$$

with the boundary condition

$$u|_{t=0^+} = \delta_{x=y} \tag{3.3}$$

admits an explicit solution

$$p^0 := \frac{k(t)}{(4\pi\Delta t)^{\nu/2}} e^{-\frac{1}{4t}A^{-1}(0)\cdot(x-y)^2+Q_t(x,y)}. \tag{3.4}$$

Here $\Delta := \det\big(A_{j,k}(0)\big)_{1\leqslant j,k\leqslant\nu}$, the function k is analytic near 0 and Q_t denotes a polynomial of total degree at most 2 in x, y whose coefficients are analytic near 0. Moreover these coefficients take their values in $i\mathbb{R}$ if $x, y \in \mathbb{R}^\nu$ and $t \in i\mathbb{R}$, $|t|$ small enough (see [Ha6] and in particular Lemma 3.4, assertion 3). By (3.3) we mean that for every $\varphi \in \mathcal{D}(\mathbb{R}^\nu)$, $x \in \mathbb{R}^\nu$ and $\theta \in [-\pi/2, \pi/2]$

$$\int_{\mathbb{R}^\nu} u(re^{i\theta}, x, y)\varphi(y)dy \longrightarrow_{r\to 0^+} \varphi(x).$$

Let us consider two particular cases. Let $\lambda \in \mathbb{R}$ and let $\omega := 2(-\lambda)^{1/2}$. Let us denote

$$p^{\mathrm{harm}} := \Big(4\pi\frac{\mathrm{sh}(\omega t)}{\omega}\Big)^{-\nu/2} \exp\Big(-\frac{1}{4}\frac{\omega}{\mathrm{sh}(\omega t)}(\mathrm{ch}(\omega t)(x^2+y^2) - 2x\cdot y)\Big)$$

and

$$p^{\mathrm{free}} = (4\pi t)^{-\nu/2} e^{-(x-y)^2/4t}.$$

Then p^{harm} (respectively p^{free}) satisfies

$$\partial_t u = \big(\partial_x^2 + \lambda x^2\big)u$$

(respectively $\partial_t u = \partial_x^2 u$) with the boundary condition (3.3).

3.2 An existence and uniqueness statement

In this section, our aim is to give an existence and uniqueness statement concerning the equation

$$\partial_t u = \big(P_0 + c(t,x)\big)u \tag{3.5}$$

We also use the following definition in this section.

Definition 3.1 *Let $T_b > 0$. Let f be a measurable $\mathbb{C}$-valued function on $D_{T_b} \times \mathbb{R}^\nu$, analytic with respect to the first variable and let μ_* be a positive measure on $\mathbb{R}^\nu$ such that for every $R > 0$*

$$\int_{\mathbb{R}^\nu} \exp(R|\xi|) \sup_{|t|<T_b} |f(t,\xi)| d\mu_*(\xi) < \infty.$$

We denote by c the function belonging to $\mathcal{A}(D_{T_b} \times \mathbb{C}^\nu)$ defined by

$$c(t,x) = \int \exp(ix\cdot\xi) f(t,\xi) d\mu_*(\xi).$$

Proposition 3.2 *Let P_0 be as above and let $T_b > 0$. There exists $T > 0$ such that, for every f and μ_* as in Definition 3.1, the following assertions hold.*

1. *For every $\varphi \in \mathcal{S}(\mathbb{R}^\nu)$ there exists a unique $\psi \in \mathcal{C}^\infty\big(i]-T,T[,\mathcal{S}(\mathbb{R}^\nu)\big)$ such that*
$$\begin{cases} \partial_t \psi = \big(P_0 + c(t,x)\big)\psi \\ \\ \psi|_{t=0} = \varphi \end{cases} \tag{3.6}$$
and the mapping $\varphi \longmapsto \psi$ is continuous from $\mathcal{S}(\mathbb{R}^\nu)$ onto $\mathcal{C}^\infty\big(i]-T,T[,\mathcal{S}(\mathbb{R}^\nu)\big)$.

2. *Let $p = p(t,x,y)$ be the kernel of the operator $\mathcal{P}_t : \mathcal{S}(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu)$ defined by $\mathcal{P}_t(\varphi) = \psi(t,\cdot)$ for $t \in i]-T,T[$. Then p can be uniquely extended as a function in $\mathcal{C}^\infty\big(\bar{D}_T^+ - \{0\}, \mathcal{A}(\mathbb{C}^{2\nu})\big) \cap \mathcal{A}\big((D_T^+ - \{0\}) \times \mathbb{C}^{2\nu}\big)$. This function satisfies (3.5) on $(\bar{D}_T^+ - \{0\}) \times \mathbb{C}^{2\nu}$ and (3.3). Moreover*
$$p = p^0 \times p^{\mathrm{conj}}$$
where $p^{\mathrm{conj}} = p^{\mathrm{conj}}(t,x,y) \in$
$$\mathcal{A}(D_T^+ \times \mathbb{C}^{2\nu}) \cap \mathcal{C}^\infty\big(\bar{D}_T^+, \mathcal{A}(\mathbb{C}^{2\nu})\big) \cap \mathcal{C}^\infty_{b,1}(i]-T,T[\times \mathbb{R}^{2\nu}).$$

3. *Let us assume that $P_0 = \partial_x^2$ (free case). Then*
$$p^{\mathrm{conj}} \in \mathcal{A}(\mathbb{C}^+ \times \mathbb{C}^{2\nu}) \cap \mathcal{C}^\infty\big(\overline{\mathbb{C}^+}, \mathcal{A}(\mathbb{C}^{2\nu})\big) \cap \mathcal{C}^\infty_b(i\mathbb{R} \times \mathbb{R}^{2\nu}).$$

Remark 3.3 *The spaces $\mathcal{C}^\infty\big(\bar{D}_T^+, \mathcal{A}(\mathbb{C}^{2\nu})\big)$ and $\mathcal{A}(D_T^+ \times \mathbb{C}^{2\nu})$ are local in the following meaning: the semi-norms are defined by taking suprema over compact sets. The spaces $\mathcal{S}(\mathbb{R}^\nu)$ and $\mathcal{C}^\infty_{b,1}(i]-T,T[\times\mathbb{R}^{2\nu})$ are global, which is useful for the uniqueness statement. By (3.4)*
$$p^0 = p^{\mathrm{free}} \times p^1$$
where $p^1 \in \mathcal{A}(D_T \times \mathbb{C}^{2\nu})$ for some $T > 0$. However one can not replace p^0 by p^{free} in the statement of Proposition 3.2 since $p^1 \notin \mathcal{C}^\infty_{b,1}(i]-T,T[\times\mathbb{R}^{2\nu})$ in the general case.

Remark 3.4 *The reality-preserving property (3.2) is useful for the uniqueness statement. This assumption implies that the operator $P_0|_{i\mathbb{R}}$ is symmetric with respect to the L^2 inner product.*

Remark 3.5 *Let us assume that $P_0 = \partial_x^2 + \lambda x^2$ where $\lambda \in \mathbb{R}$, that the potential c does not depend on t. Since $c \in L^\infty(\mathbb{R}^\nu)$, by [Re-Si, Th. X.36], the operator $P_0 + c$ is self-adjoint on a domain containing the domain of the operator $-\partial_x^2 + x^2$. See also [Bo-Ca-Hä-Mi]. Therefore one can define the Schrödinger operator for the operator P_0. By [Ha4], its kernel admits a (unique) analytic continuation on $(\bar{D}_T^+ - \{0\}) \times \mathbb{C}^{2\nu}$ if T is small enough. This yields an alternative formulation of Proposition 3.2 in the harmonic case.*

3.3 Proof of Proposition 3.2

We need the following lemma.

Lemma 3.6 *Let $T > 0$ and let B be a $\nu \times \nu$ real positive definite symmetric matrix. Let $P_{\boldsymbol{t}}$ be a polynomial with respect to $x, y \in \mathbb{R}^\nu$, of degree at most 2, with coefficients belonging to $\mathcal{C}^\infty(]-T,T[,\mathbb{R})$. There exists $T_2 > 0$ such that, for every $u \in \mathcal{C}^\infty_{b,1}(]-T,T[\times\mathbb{R}^{2\nu})$ and every $\varphi \in \mathcal{S}(\mathbb{R}^\nu)$, the function $\boldsymbol{\psi}$ defined by*

$$\boldsymbol{\psi}(\boldsymbol{t},x) := \int_{\mathbb{R}^\nu} \Big(\frac{\det B}{4\pi i\boldsymbol{t}}\Big)^{\nu/2} e^{-B\cdot(x-y)^2/4i\boldsymbol{t}} e^{iP_{\boldsymbol{t}}(x,y)} u(\boldsymbol{t},x,y)\varphi(y)dy \tag{3.7}$$

belongs to $\mathcal{C}^\infty\big(]-T_2,T_2[,\mathcal{S}(\mathbb{R}^\nu)\big)$. The mapping $\varphi \longmapsto \boldsymbol{\psi}$ is continuous from $\mathcal{S}(\mathbb{R}^\nu)$ into $\mathcal{C}^\infty\big(]-T_2,T_2[,\mathcal{S}(\mathbb{R}^\nu)\big)$. Moreover, if $e^{iP_0(y,y)} \times u(0,y,y) = 1$, then $\boldsymbol{\psi}|_{\boldsymbol{t}=0} = \varphi$.

Such a result is standard. The proof is given in the Appendix.

Let us prove Proposition 3.2. The assertion 1, of course, is a well-known statement. However, for the convenience of the reader and the completeness of the paper, we give its proof.

-1- In view of the uniqueness statement in assertion 1, let us consider $\psi \in \mathcal{C}^\infty\big(i]-T,T[,\mathcal{S}(\mathbb{R}^\nu)\big)$ satisfying (3.6) with $\varphi = 0$. Let

$$E(\boldsymbol{t}) := \int_{\mathbb{R}^\nu} \psi(i\boldsymbol{t},x)\bar{\psi}(i\boldsymbol{t},x)dx,$$

$$R(\boldsymbol{t}) := -2\int_{\mathbb{R}^\nu} \mathcal{I}m\big(c(i\boldsymbol{t},x)\big)\psi(i\boldsymbol{t},x)\bar{\psi}(i\boldsymbol{t},x)dx.$$

Then by (3.2), for every $\boldsymbol{t} \in]-T,T[$, $\partial_{\boldsymbol{t}}E = R(\boldsymbol{t})$. Since the function $|c|$ is bounded, there exists $K > 0$ such that $|\partial_{\boldsymbol{t}}E| \leqslant KE$. Since $\psi\bar{\psi}$ is bounded by $C(1+|x|)^{-\nu-1}$ for $(\boldsymbol{t},x) \in]-T,T[\times\mathbb{R}^\nu$, one gets $E|_{\boldsymbol{t}=0} = 0$ by the dominated convergence theorem. Therefore $E = 0$ and $\psi|_{i]-T,T[\times\mathbb{R}^\nu} \equiv 0$.

-2- Let us prove the existence statement in assertion 1. Let p^{conj} be as in [Ha6, Theorem 2.1]. Let $\varphi \in \mathcal{S}(\mathbb{R}^\nu)$. Let

$$\psi(t,x) := \int_{\mathbb{R}^\nu} (p^0 \times p^{\mathrm{conj}})(t,x,y)\varphi(y)dy.$$

By (3.4)

$$(p^0 \times p^{\mathrm{conj}})(i\boldsymbol{t},x,y) = (4\pi i\boldsymbol{\Delta t})^{-\nu/2} e^{-A^{-1}(0)\cdot(x-y)^2/4i\boldsymbol{t}} \times e^{iP_{\boldsymbol{t}}} \times k(i\boldsymbol{t})p^{\mathrm{conj}}(i\boldsymbol{t},x,y)$$

where the polynomial $P_{\boldsymbol{t}}$ satisfies the assumptions of Lemma 3.6. Then, by Lemma 3.6, there exists $T_2 \in]0,T]$ such that $\psi \in \mathcal{C}^\infty\big(i]-T_2,T_2[,\mathcal{S}(\mathbb{R}^\nu)\big)$. Moreover the mapping $\varphi \longmapsto \psi$ is continuous from $\mathcal{S}(\mathbb{R}^\nu)$ onto $\mathcal{C}^\infty\big(i]-T,T[,\mathcal{S}(\mathbb{R}^\nu)\big)$. Since $p^0 \times p^{\mathrm{conj}}$ satisfies (3.5) and $p^{\mathrm{conj}}|_{t=0} = 1$, (3.6) holds.

-3- Let us prove assertion 2. By the regularity properties of p^{conj} [Ha6, Theorem 2.1], one gets a suitable extension of the kernel of the operator $\mathcal{P}_t$. We claim that this extension is unique. Let p_1 and p_2 be two extensions. Then

$$p = p_1 - p_2 \in \mathcal{A}\big((D_T^+ - \{0\}) \times \mathbb{C}^{2\nu}\big) \cap \mathcal{C}^\infty\big(\bar{D}_T^+ - \{0\}, \mathcal{A}(\mathbb{C}^{2\nu})\big)$$

and $p|_{(i]-T,T[-\{0\})\times\mathbb{R}^\nu} = 0$. Let $(x,y) \in \mathbb{R}^{2\nu}$ and let $\breve{p}$ be the function on $D_T - \{0\}$ defined by $\breve{p}(t) := 1_{\mathcal{R}et\geqslant 0}\, p(t,x,y)$. By regularity properties of p and Cauchy-Riemann equations with respect to t, $\breve{p}$ is smooth near $iT/2$ and satisfies Cauchy-Riemann equations. Therefore the function $\breve{p}$ is analytic near $iT/2$, vanishes near $iT/2$ and actually on $D_T^+ - \{0\}$. Then by analytic continuation with respect to the space variables the function p vanishes on $(D_T^+ - \{0\}) \times \mathbb{C}^{2\nu}$.

-4- Assertion 3 can also be checked by considering the deformation formula in the free case.

4 The heat kernel viewed from an arbitrary direction

We must take into account the ramification of the heat kernel at $t = 0$ in our statements. The ramification is only due to the term $t^{-\nu/2}$ in (3.4). Let $\epsilon \in \mathbb{R}/4\pi\mathbb{Z}$. Then the free heat kernel p^{free} is invariant, up to a multiplicative constant, under the change of variables

$$(t,x,y) \mapsto (e^{i\epsilon}t, e^{i\epsilon/2}x, e^{i\epsilon/2}y) \tag{4.1}$$

and the free heat equation is invariant under the change of variables $(t,x) \mapsto (e^{i\epsilon}t, e^{i\epsilon/2}x)$. This elementary remark allows a reformulation of Proposition 3.2 (we only consider the harmonic case for the sake of simplicity). We denote[4]

$$p_\epsilon^{\mathrm{harm}} := \big(4\pi e^{-i\epsilon}t\big)^{-\nu/2} \times$$

$$\left(\frac{\mathrm{sh}(\omega e^{-i\epsilon}t)}{\omega e^{-i\epsilon}t}\right)^{-\nu/2} \exp\Big(-\frac{1}{4}\frac{e^{-i\boldsymbol{\pi}(\epsilon)}\omega}{\mathrm{sh}(\omega e^{-i\epsilon}t)}(\mathrm{ch}(\omega e^{-i\epsilon}t)(x^2+y^2) - 2x\cdot y)\Big)\ , \quad |t| \ll 1\ ,$$

$$p_\epsilon^{\mathrm{free}} := \big(4\pi e^{-i\epsilon}t\big)^{-\nu/2} \exp\Big(-\frac{1}{4t}(x-y)^2\Big).$$

Corollary 4.1 *Let $\lambda \in \mathbb{R}$ and $\epsilon \in \mathbb{R}/4\pi\mathbb{Z}$. There exists $T > 0$ such that the following statement holds. Let μ be a complex measure on $\mathbb{R}^\nu$ such that for every $R > 0$*

$$\int_{\mathbb{R}^\nu} \exp(R|\xi|)d|\mu|(\xi) < \infty. \tag{4.2}$$

[4]Only the first factor of the product defining $p_\epsilon^{\mathrm{harm}}$ is concerned by the ramification.

Let

$$c(x) = \int_{\mathbb{R}^\nu} \exp(ie^{-i\epsilon/2} x \cdot \xi) d\mu(\xi).$$

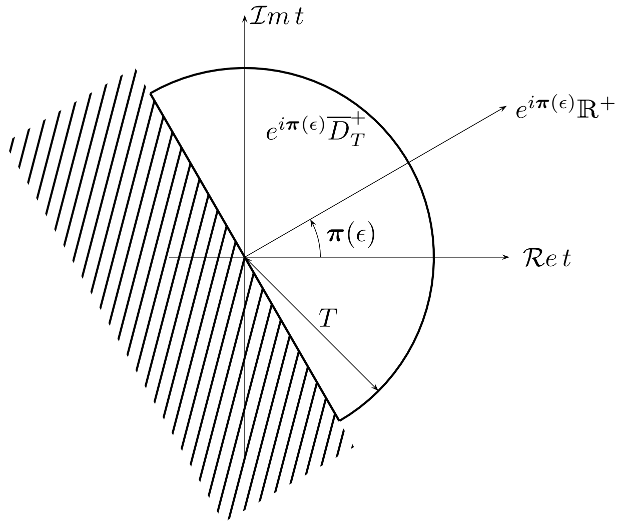
Then the following assertions hold.

1. *For every* $\varphi \in \mathcal{S}(e^{i\epsilon/2}\mathbb{R}^\nu)$ *there exists a unique* $\psi \in \mathcal{C}^\infty\big(ie^{i\epsilon}]-T,T[, \mathcal{S}(e^{i\epsilon/2}\mathbb{R}^\nu)\big)$ *such that*
$$\begin{cases} \partial_t \psi = \big(\partial_x^2 + \lambda e^{-2i\boldsymbol{\pi}(\epsilon)} x^2 + c(x)\big)\psi \\ \\ \psi|_{t=0} = \varphi \end{cases}$$
and the mapping $\varphi \longmapsto \psi$ *is continuous from* $\mathcal{S}(e^{i\epsilon/2}\mathbb{R}^\nu)$ *onto* $\mathcal{C}^\infty\big(ie^{i\epsilon}] - T, T[, \mathcal{S}(e^{i\epsilon/2}\mathbb{R}^\nu)\big)$.

2. *Let* $p = p(t,x,y)$ *be the kernel of the operator* $\mathcal{P}_t : \mathcal{S}(e^{i\epsilon/2}\mathbb{R}^\nu) \to \mathcal{S}(e^{i\epsilon/2}\mathbb{R}^\nu)$ *defined by* $\mathcal{P}_t(\varphi) = \psi(t,\cdot)$ *for* $t \in ie^{i\epsilon}] - T, T[$. *Then* p *can be uniquely continued as a function belonging to*
$$\mathcal{A}\big(e^{i\epsilon}(D_T^+ - \{0\}) \times \mathbb{C}^{2\nu}\big) \cap \mathcal{C}^\infty\big(e^{i\epsilon}(\bar{D}_T^+ - \{0\}), \mathcal{A}(\mathbb{C}^{2\nu})\big).$$
Moreover
$$p = p_\epsilon^{\mathrm{harm}} \times p^{\mathrm{conj}}$$
where $p^{\mathrm{conj}} = p^{\mathrm{conj}}(t,x,y) \in$
$$\mathcal{A}(e^{i\boldsymbol{\pi}(\epsilon)} D_T^+ \times \mathbb{C}^{2\nu}) \cap \mathcal{C}^\infty\big(e^{i\boldsymbol{\pi}(\epsilon)} \bar{D}_T^+, \mathcal{A}(\mathbb{C}^{2\nu})\big) \cap \mathcal{C}_{b,1}^\infty(ie^{i\boldsymbol{\pi}(\epsilon)}] - T, T[\times \mathbb{R}^{2\nu}).$$

3. *Let us assume that* $\lambda = 0$. *Then*
$$p^{\mathrm{conj}} \in \mathcal{A}(e^{i\boldsymbol{\pi}(\epsilon)}\mathbb{C}^+ \times \mathbb{C}^{2\nu}) \cap \mathcal{C}^\infty\big(e^{i\boldsymbol{\pi}(\epsilon)}\overline{\mathbb{C}^+}, \mathcal{A}(\mathbb{C}^{2\nu})\big) \cap \mathcal{C}_b^\infty(ie^{i\boldsymbol{\pi}(\epsilon)}\mathbb{R} \times \mathbb{R}^{2\nu}).$$

Figure 4.1:



Remark 4.2 *Let $\epsilon \in \mathbb{R}/4\pi\mathbb{Z}$. Let $f = f(t,x,y)$ be a continuous function on $\left(e^{i\epsilon}(\bar{D}_T^+ - \{0\})\right) \times \mathbb{C}^\nu \times \mathbb{C}^\nu$. We say that f goes to $\delta_{x=y}$ in the direction $e^{i\epsilon}$ and write*

$$f|_{t=e^{i\epsilon}0^+} = \delta_{x=y}$$

if and only if for every $\theta, \vartheta \in \mathbb{R}/4\pi\mathbb{Z}$, $|\theta - \epsilon| \leqslant \pi/2$, $|\theta - 2\vartheta| \leqslant \pi/2$ and every $\varphi \in \mathcal{D}(e^{i\vartheta}\mathbb{R}^\nu)$, $x \in e^{i\vartheta}\mathbb{R}^\nu$

$$\int_{e^{i\vartheta}\mathbb{R}^\nu} f(re^{i\theta}, x, y)\varphi(y)dy \longrightarrow_{r\to 0^+} \varphi(x).$$

Here $dy = e^{i\vartheta\nu}dm(y)$ where m denotes the standard (non negative) Lebesgue measure on $e^{i\vartheta}\mathbb{R}^\nu$. Then the kernel p satisfies on $e^{i\epsilon}(D_T^+ - \{0\}) \times \mathbb{C}^{2\nu}$

$$\begin{cases} \partial_t p = \left(\partial_x^2 + \lambda e^{-2i\boldsymbol{\pi}(\epsilon)}x^2 + c(x)\right)p \\ \\ p|_{t=e^{i\epsilon}0^+} = \delta_{x=y} \end{cases} \tag{4.3}$$

since, for every smooth function $g = g(t,x,y)$ on $e^{i\boldsymbol{\pi}(\epsilon)}\bar{D}_T^+ \times \mathbb{C}^\nu \times \mathbb{C}^\nu$ such that $g|_{t=0} = 1$, the function $p_\epsilon^{\text{free}} \times g$ goes to $\delta_{x=y}$ in the direction $e^{i\epsilon}$.

Let us choose $\epsilon = \pi$. Then we get a solution p such that p^{conj} is defined on

$$\left\{t = |t|e^{i\theta} \in \mathbb{C} | \theta \in [\pi/2, 3\pi/2]_{4\pi}, |t| < T\right\} \times \mathbb{C}^{2\nu}$$

or $\left\{t \in \mathbb{C} | \theta \in [\pi/2, 3\pi/2]_{4\pi}\right\} \times \mathbb{C}^{2\nu}$ if $\lambda = 0$. In particular, by considering values of t such that $\arg t = \pi/2, 3\pi/2$, we obtain the following result about the standard Schrödinger equation.

Corollary 4.3 *Let μ as in Corollary 4.1. Let*

$$c(x) = \int_{\mathbb{R}^\nu} \exp(x \cdot \xi) d\mu(\xi). \tag{4.4}$$

Let $\lambda \in \mathbb{R}$. Then there exist $T > 0$ and

$$p^{\mathrm{conj}} = p^{\mathrm{conj}}(\boldsymbol{t}, x, y) \in \mathcal{C}^\infty\big(]-T, T[, \mathcal{A}(\mathbb{C}^{2\nu})\big)$$

such that $p = p_\epsilon^{\mathrm{harm}} \times p^{\mathrm{conj}}$ satisfies

$$\begin{cases} \frac{1}{i}\partial_{\boldsymbol{t}} p = \big(\partial_x^2 + \lambda x^2 + c(x)\big)p \ , \quad x \in \mathbb{R}^\nu, \boldsymbol{t} \in]-T, T[\\ \\ p|_{\boldsymbol{t}=0} = \delta_{x=y} \ , \quad y \in \mathbb{R}^\nu \end{cases} . \tag{4.5}$$

If $\lambda = 0$, $p^{\mathrm{conj}} \in \mathcal{C}^\infty\big(\mathbb{R}, \mathcal{A}(\mathbb{C}^{2\nu})\big)$.

Remark 4.4 *The assumption (4.2), since c is given by (4.4), allows potentials such as*

$$V(x) = \lambda x^2 \pm e^{x^2}, V(x) = \lambda x^2 \pm e^{x_1}, \ldots$$

In the case $\lambda = 0$, this fact was noticed by Kuna, Streit and Westerkamp [K-S-W]. In particular the function $x \longmapsto e^{x^2}$ is viewed as a perturbation of the operator $\partial_x^2 + \lambda x^2$ (the deformation formula is used to deal with this part of the potential V) whereas the function $x \longmapsto \lambda x^2$ is not viewed as a perturbation of the operator ∂_x^2 in our method. Using a complex point of view with respect to the space variables perhaps explains this "paradox". A related remark can be done for the uniqueness problem: we do not claim that (4.5) has a unique solution in its natural real setting. However it does, if we consider complex values for the space variables ($x, y \in e^{i\pi/2}\mathbb{R}^\nu$), by taking advantage of the uniqueness statement of Corollary 4.1.

Notice that the dilation given by $(t, x, y) \mapsto (e^{i\pi}t, ix, iy)$ ($\epsilon = \pi$ in 4.1), which allows one to view Corollary 4.3 as a consequence of Corollary 4.1, "reverses" the direction of t.

Another viewpoint is formally related to the previous proposition. For $\theta > 0$, let

$$\mathbb{R}^{+,\nu}_{\prec,\theta} := \{e^{i\varphi}x \in \mathbb{C}^\nu | x \in (\mathbb{R}^+)^\nu, \varphi \in]-\theta, \theta[_{2\pi}\},$$

$$\mathbb{R}^{\nu}_{\prec,\theta} := \{e^{i\varphi}x \in \mathbb{C}^\nu | x \in \mathbb{R}^\nu, \varphi \in]-\theta, \theta[_{2\pi}\},$$

$$\mathbb{R}^{+}_{\prec,\theta} := \{re^{i\varphi} \in \mathbb{C} | r > 0, \varphi \in]-\theta, \theta[_{2\pi}\}.$$

One has

Proposition 4.5 *Let $\theta, \alpha \in]0, \pi/4[$. Let μ be a $\mathbb{C}$-valued Borel measure on $\mathbb{C}^\nu$. Let us assume (case 1) that $\mathbb{R}^{+,\nu}_{\prec,\theta}$ contains the support of μ and that*

$$\forall R > 0, \int_{\mathbb{C}^\nu} \exp(R|\xi|) d|\mu|(\xi) < \infty$$

or (case 2) that $d\mu(\xi) = \hat{c}(\xi)d\xi$ where $\hat{c}$ denotes an analytic function on $\mathbb{R}^\nu_{\prec,\alpha}$ satisfying

$$\forall R > 0, \exists K > 0, \forall \xi \in \mathbb{R}^\nu_{\prec,\alpha}, |\hat{c}(\xi)| \leqslant K e^{-R|\xi|}.$$

Let $F = \mathbb{C}^\nu$ (case 1) or $F = \mathbb{R}^\nu$ (case 2). Let

$$c(x) := \int_F \exp(ix \cdot \xi) d\mu(\xi).$$

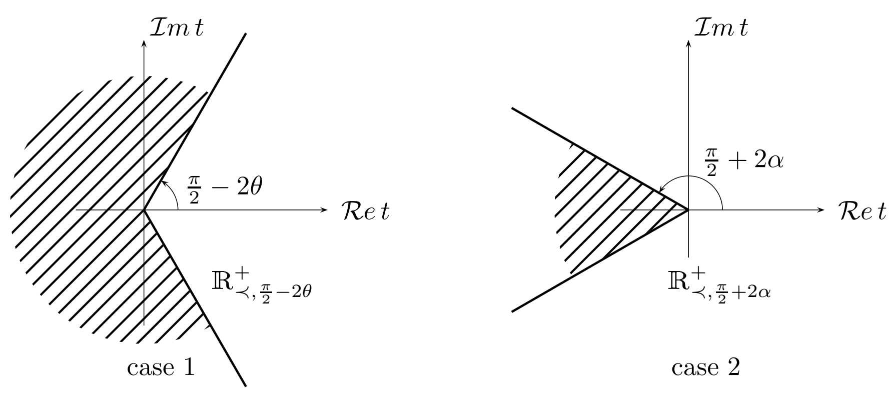
Then, by the deformation formula,

- *case 1: The heat equation associated to the operator $\partial_x^2 + c(x)$ has a solution p defined on $\left(\mathbb{R}^+_{\prec,\pi/2-2\theta} - \{0\}\right) \times \mathbb{C}^{2\nu}$ satisfying the following boundary condition. For every $\varphi \in \mathcal{D}(\mathbb{R}^\nu)$, $x \in \mathbb{R}^\nu$ and $\alpha \in]-\pi/2 + 2\theta, \pi/2 - 2\theta[$*

$$\int_{\mathbb{R}^\nu} p(re^{i\alpha}, x, y)\varphi(y)dy \longrightarrow_{r \to 0^+} \varphi(x).$$

- *case 2: The heat kernel of the operator $\partial_x^2 + c(x)$, which is defined on $\mathbb{R}^+ \times \mathbb{R}^{2\nu}$, admits an analytic continuation on $\left(\mathbb{R}^+_{\prec,\pi/2+2\alpha} - \{0\}\right) \times \mathbb{C}^{2\nu}$.*

Figure 4.2:



Proof Let $p^{\mathrm{conj}} := \sum_{n \geqslant 0} v_n$ where

$$v_n := t^n \int_{0<s_1<\cdots<s_n<1} \int_F e^{i\left(y+s(x-y)\right)\cdot\xi} \exp\left(-ts(1-s) \cdot_n \xi \otimes \xi\right) d^{\nu n}\mu^{\otimes}(\xi) d^n s,$$

$$\left(y + s(x-y)\right) \cdot \xi := \left(y + s_1(x-y)\right) \cdot \xi_1 + \cdots + \left(y + s_n(x-y)\right) \cdot \xi_n,$$

$$s(1-s) \cdot_n \xi \otimes \xi := \sum_{j,k=1}^n s_{j\wedge k}(1 - s_{j\vee k})\xi_j \cdot \xi_k,$$

$$d^{\nu n}\mu^{\otimes}(\xi) := d\mu(\xi_n)\cdots d\mu(\xi_1).$$

We first check that the series defining p^{conj} is convergent.

(case 1) We claim that $p^{\text{conj}} = p^{\text{conj}}(t,x,y) \in \mathcal{A}(\mathbb{R}^+_{\prec,\pi/2-2\theta} \times \mathbb{C}^{2\nu})$. One has

$$|x|,|y| \leqslant R \Rightarrow \big|\exp\big(i\big(y+s(x-y)\big)\cdot\xi\big)\big| \leqslant e^{R|\xi_1|}\times\cdots\times e^{R|\xi_n|}.$$

Since $2\theta < \pi/2$, $\mathbb{R}^+_{\prec,2\theta}$ is a convex cone. Then

$$t \in \mathbb{R}^+_{\prec,\pi/2-2\theta}, \xi \in \operatorname{supp}(\mu^{\otimes}) \Rightarrow \mathcal{R}e\big(ts(1-s)\cdot_n \xi\otimes\xi\big) \geqslant 0.$$

This implies the convergence of the series defining p^{conj} and the analyticity of p^{conj}.

(case 2) We claim that $p^{\text{conj}} \in \mathcal{A}(\mathbb{R}^+_{\prec,\pi/2+2\alpha} \times \mathbb{C}^{2\nu})$. Let $\beta \in]0,\alpha[$. Since the function $\hat{c}$ is analytic on $\mathbb{R}^{\nu}_{\prec,\alpha}$, one gets by a deformation of the integration contour,

$$v_n := e^{-i\nu n\beta} t^n \int_{0<s_1<\cdots<s_n<1} \int_{\mathbb{R}^{\nu n}} e^{ie^{-i\beta}\big(y+s(x-y)\big)\cdot\xi}$$
$$\exp\big(-te^{-2i\beta}s(1-s)\cdot_n \xi\otimes\xi\big)\hat{c}(e^{-i\beta}\xi_1)\cdots\hat{c}(e^{-i\beta}\xi_n)d^{\nu n}\xi d^n s.$$

Therefore the convergence of the series defining p^{conj} and the analyticity of p^{conj} hold for $\mathcal{R}e\big(e^{-2i\beta}t\big) > 0$ and $x,y \in \mathbb{C}^{\nu}$. Since β is arbitrary, one gets that $p^{\text{conj}} \in \mathcal{A}(\mathbb{R}^+_{\prec,\pi/2+2\alpha} \times \mathbb{C}^{2\nu})$.

By proceeding as in [Ha4], one can show that $p = p^{\text{free}} \times p^{\text{conj}}$ satisfies the heat equation. Moreover the boundary condition is satisfied. □

Example 4.6 *Let $\theta_1,\ldots,\theta_q \in]-\pi/4,\pi/4[$ and $\lambda_1,\ldots,\lambda_q \in (\mathbb{R}^+)^{\nu}$. Let*

$$c(x) = \exp\big(ie^{i\theta_1}\lambda_1\cdot x\big) + \cdots + \exp\big(ie^{i\theta_q}\lambda_q\cdot x\big).$$

Then the function c satisfies the assumptions of Proposition 4.5 (case 1). We do not attempt to give a uniqueness statement in this case.

Example 4.7 *Let $c(x) = e^{-x^2}$. By Proposition 4.5 (case 2) the heat kernel is well defined on $(\mathbb{C}-]-\infty,0]) \times \mathbb{C}^{2\nu}$.*

Remark 4.8 *One can generalize Proposition 4.5 (case 2) in the harmonic case. The proof needs a modification of [Ha4, Lemma 4.2].*

5 Borel summability of the conjugate of the heat kernel in an arbitrary direction

For the sake of simplicity, we only consider the free case in this section. Let $\kappa, T > 0$. Let

$$\tilde{S}_\kappa := \big\{z \in \mathbb{C} | d(z,[0,+\infty[) < \kappa\big\}\ ,\ \ \tilde{D}_T := \Big\{z \in \mathbb{C} | \mathcal{R}e\big(\frac{1}{z}\big) > \frac{1}{T}\Big\}.$$

$\tilde{D}_T$ is the open disk of center $\frac{T}{2}$ and radius $\frac{T}{2}$.

Definition 5.1 *Let $\dot{\epsilon} \in \mathbb{R}/2\pi\mathbb{Z}$. Let $a_1, \ldots, a_r, \ldots \in \mathbb{C}$. The formal power series $\tilde{f} = \sum_{r \geqslant 0} a_r t^r$ is called Borel-Nevalinna (respectively Borel-Watson) summable in the direction $e^{i\dot{\epsilon}}$ if*

- *the radius of convergence of the Borel transform of $\tilde{f}$, $\hat{f}(\tau) := \sum_{r=0}^{\infty} \frac{a_r}{r!} \tau^r$, does not vanish*
- *there exist $\kappa > 0$ (respectively $\theta > 0$) such that the Borel transform can be analytically continued on $e^{i\dot{\epsilon}} \tilde{S}_\kappa$ (respectively $e^{i\dot{\epsilon}} \mathbb{R}^+_{\prec,\theta}$)*
- *there exist $K, T > 0$ such that for every $\tau \in e^{i\dot{\epsilon}} \tilde{S}_\kappa$ (respectively $e^{i\dot{\epsilon}} \mathbb{R}^+_{\prec,\theta}$)*

$$|\hat{f}(\tau)| \leqslant K e^{|\tau|/T}.$$

If the power series $\tilde{f}$ is Borel-Nevalinna or Borel-Watson summable, the Laplace transform of $\hat{f}$

$$f(t) := \int_0^{+\infty} \hat{f}(\tau) e^{-\frac{\tau}{t}} \frac{d\tau}{t}$$

is called the Borel sum of $\tilde{f}$.

Figure 5.1:

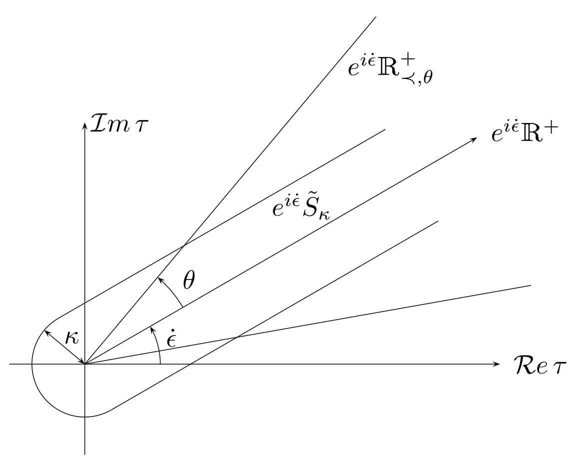
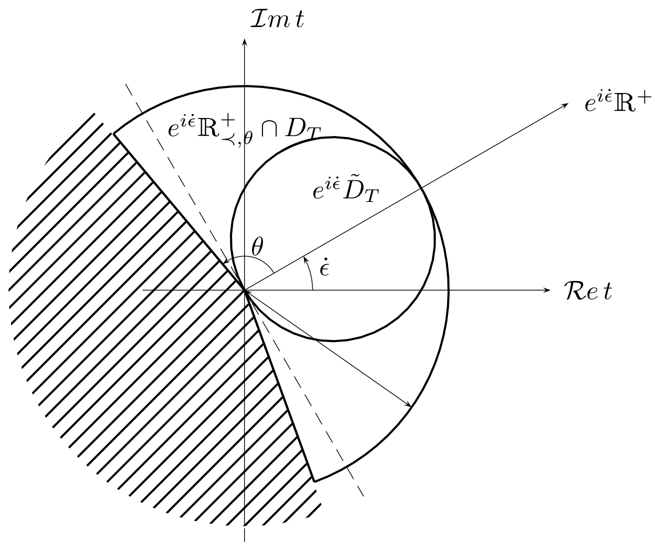


Figure 5.2:



Remark 5.2 *If a power series $\tilde{f}$ is Borel-Nevalinna (respectively Borel-Watson) summable in the direction $e^{i\dot\epsilon}$, then there exist $T > 0$ and $\theta > \pi/2$ such that its Borel sum is well defined for $t \in e^{i\dot\epsilon}\tilde{D}_T$ (respectively $e^{i\dot\epsilon}\mathbb{R}^+_{\prec,\theta} \cap D_T$). See [So].*

The change of variables (4.1) allows us to give the following corollary of Theorem 3.1 [Ha4].

Corollary 5.3 *Let $\varepsilon > 0$ and $\epsilon \in \mathbb{R}/4\pi\mathbb{Z}$. Let μ be a $\mathbb{C}$-valued measure on $\mathbb{R}^\nu$ verifying*

$$\int_{\mathbb{R}^\nu} \exp(\varepsilon\xi^2) d|\mu|(\xi) < \infty. \tag{5.1}$$

Let

$$c(x) = \int \exp(ie^{-i\epsilon/2} x \cdot \xi) d\mu(\xi) \tag{5.2}$$

and let u be the solution of (4.3) where $\lambda = 0$. Let p^{conj} be defined by $u = p^{\text{free}} p^{\text{conj}}$. Then p^{conj} admits a Borel transform $\widehat{p^{\text{conj}}}$ (with respect to t) which is analytic on $\mathbb{C}^{1+2\nu}$. Let $\kappa, R > 0$ and let

$$C := 2\Big(\int \exp\big(\frac{2\kappa}{\varepsilon} + \frac{\varepsilon}{2}\xi^2 + R|\xi|\big) d|\mu|(\xi)\Big)^{1/2}.$$

Then, for every $(\tau, x, y) \in e^{i\boldsymbol{\pi}(\epsilon)}\tilde{S}_\kappa \times \mathbb{C}^{2\nu}$ such that $\big|\mathcal{I}m(e^{-i\epsilon/2}x)\big| < R$ and $\big|\mathcal{I}m(e^{-i\epsilon/2}y)\big| < R$,

$$\big|\widehat{p^{\text{conj}}}(\tau, x, y)\big| \leqslant \exp\big(C|\tau|^{1/2}\big). \tag{5.3}$$

Remark 5.4 *By the estimate (5.3), the small time expansion of the conjugate heat kernel is Borel-Nevalinna summable in the direction* $e^{i\boldsymbol{\pi}(\epsilon)}$ *and its Borel sum is equal to* p^{conj}.

We now illustrate Corollary 5.3 by simple examples.

Example 5.5 *Let* $\epsilon \in \mathbb{R}/4\pi\mathbb{Z}$, $\xi_0 \in e^{-i\epsilon/2}\mathbb{R}^\nu - \{0\}$ *and* $c(x) = \exp(ix\cdot\xi_0)$. *The function c satisfies the assumptions of Corollary 5.3 hence the small time expansion of the conjugate heat kernel is Borel-Nevalinna summable in the direction* $e^{i\boldsymbol{\pi}(\epsilon)}$. *For* $\epsilon' \in \mathbb{R}/4\pi\mathbb{Z}$, $\boldsymbol{\pi}(\epsilon') \neq \boldsymbol{\pi}(\epsilon)$, *the function c is not bounded on* $e^{i\epsilon'/2}\mathbb{R}^\nu$ *and therefore does not satisfy the assumptions of Corollary 5.3 in the direction* $e^{i\varepsilon'}$*: Corollary 5.3 can not be used to study the Borel-Watson summability of this expansion in the direction* $e^{i\boldsymbol{\pi}(\epsilon)}$.

Example 5.6 *Let* $c(x) = \exp\big(ix_1 + ie^{i\pi/8}x_2\big)$ *and let p be the solution given by Proposition 4.5 (case* 1*). Then Corollary 5.3 can not help us to study the Borel summability of the small time expansion of p.*

Let us now consider $c(x) = \exp\big(ix_1\big) + \exp\big(ie^{i\pi/8}x_2\big)$. *Then by separation of variables, the solution given by Proposition 4.5 is the product of two Borel-Nevalinna summable expansions but in different directions.*

Example 5.7 *Let* $K, A > 0$ *and* $\alpha \in]0, \pi/2[$. *Let* $\hat{c}$ *be an analytical function on* $\mathbb{R}^\nu_{\prec,\alpha}$ *satisfying, for every* $\xi \in \mathbb{R}^\nu_{\prec,\alpha}$,

$$|\hat{c}(\xi)| \leqslant Ke^{-A|\xi|^2}.$$

Let

$$c(x) := \int_{\mathbb{R}^\nu} \exp(ix\cdot\xi)\hat{c}(\xi)d\xi.$$

Let $\epsilon \in I :=]-2\alpha, 2\alpha[_{4\pi}$ *and* $\varepsilon < A$. *Then there exists a measure* μ *satisfying (5.1) such that the function c is also defined by (5.2) (see also Proposition 4.5 case* 2*). Therefore the small time expansion of the conjugate heat kernel is Borel-Nevalinna or Borel-Watson summable in every direction belonging to* $\boldsymbol{\pi}(I)$. *Functions like* $c = e^{-\gamma x^2}$, $\gamma > 0$, *satisfy such a property.*

6 Appendix

Here is a proof of Lemma 3.6.

For the sake of simplicity, we assume $B = \mathrm{Id}$. For $\delta \in \mathbb{N}^\nu$, we denote $|\delta| = \delta_1 + \cdots + \delta_\nu$. For $m, k \in \mathbb{R}$ and $p \in \mathbb{N}$, we denote by

$$S^{m,k}_p(]-T, T[\times\mathbb{R}^\nu\times\mathbb{R}^\nu)$$

the set of smooth functions $f = f(\boldsymbol{t}, x, y)$ on $]-T, T[\times\mathbb{R}^\nu\times\mathbb{R}^\nu$ such that

$$\forall(q,r)\in\mathbb{N}^2, \exists C>0, \forall(\alpha,\beta,\gamma)\in\mathbb{N}\times\mathbb{N}^\nu\times\mathbb{N}^\nu, \forall(\boldsymbol{t},x,y)\in]-T,T[\times\mathbb{R}^\nu\times\mathbb{R},$$

$$\alpha \leqslant p, |\beta| \leqslant q, |\gamma| \leqslant r \Rightarrow |\partial_{\boldsymbol{t}}^{\alpha}\partial_x^{\beta}\partial_y^{\gamma} f| \leqslant C(1+|x|)^{m+\alpha}(1+|y|)^{k+\alpha}.$$

For such a function, we denote by $|f|_{m,k,p,q,r}$ the best constant C satisfying the previous inequality. For $f \in S_p^{m,k}$ $(k < -\nu)$, let us denote

$$\mathcal{F}f(\boldsymbol{t}, x) := \int_{\mathbb{R}^\nu} (4\pi i\boldsymbol{t})^{-\nu/2} e^{-(x-y)^2/4i\boldsymbol{t}} e^{iP_{\boldsymbol{t}}(x,y)} f(\boldsymbol{t}, x, y) dy.$$

We first establish some useful properties of this transform. Let $f \in S_r^{m,k}$. Let $j = 1, \dots, \nu$.

- Using the symmetry of the free Schrödinger kernel and integration by parts, one gets
$$\partial_{x_j} \mathcal{F}f = \mathcal{F}\tilde{f} \tag{6.1}$$
where
$$\tilde{f} := e^{-iP_{\boldsymbol{t}}(x,y)}(\partial_{x_j} + \partial_{y_j})\big(e^{iP_{\boldsymbol{t}}(x,y)} f(\boldsymbol{t}, x, y)\big) \in S_{r-1}^{m+1,k+1}.$$
Moreover there exists $c_2 > 0$, which only depends on the coefficients of $P_{\boldsymbol{t}}(x, y)$, such that
$$|\tilde{f}|_{m+1,k+1,p,q-1,r-1} \leqslant c_2 |f|_{m,k,p,q,r}.$$

- Since
$$\frac{1}{i}\partial_{\boldsymbol{t}}\big((4\pi i\boldsymbol{t})^{-\nu/2} e^{-(x-y)^2/4i\boldsymbol{t}}\big) = \partial_y^2\big((4\pi i\boldsymbol{t})^{-\nu/2} e^{-(x-y)^2/4i\boldsymbol{t}}\big)$$
and by integrations by parts, one gets
$$\partial_{\boldsymbol{t}} \mathcal{F}f = \mathcal{F}\tilde{f} \tag{6.2}$$
where
$$\tilde{f} := e^{-iP_{\boldsymbol{t}}(x,y)}(i\partial_y^2 + \partial_{\boldsymbol{t}})\big(e^{iP_{\boldsymbol{t}}(x,y)} f(\boldsymbol{t}, x, y)\big) \in S_{r-2}^{m+2,k+2}.$$
Moreover there exists $c_1 > 0$ such that
$$|\tilde{f}|_{m+2,k+2,p-1,q,r-2} \leqslant c_1 |f|_{m,k,p,q,r}.$$

- We shall need to estimate $x_j \mathcal{F}f$. For this, we express the multiplication operator by x_j in a convenient way. Let us denote
$$\phi := \frac{(x-y)^2}{4\boldsymbol{t}} + P_{\boldsymbol{t}}(x, y).$$
Let $\varsigma = 1, \dots, \nu$. Then
$$\partial_{y_\varsigma}\phi = \frac{y_\varsigma - x_\varsigma}{2\boldsymbol{t}} + \frac{1}{2}c(\boldsymbol{t}) + \frac{1}{2}\sum_{\varsigma'=1}^{\nu}\big(a_{\varsigma,\varsigma'}(\boldsymbol{t})x_{\varsigma'} + b_{\varsigma,\varsigma'}(\boldsymbol{t})y_{\varsigma'}\big)$$

where $a_{\varsigma,\varsigma'}, b_{\varsigma,\varsigma'}, c$ are smooth $\mathbb{R}$-valued functions defined on $]-T,T[$. Then

$$\big(1-\boldsymbol{t}a_{\varsigma,\varsigma}(\boldsymbol{t})\big)x_\varsigma-\boldsymbol{t}\sum_{\substack{\varsigma'=1\\ \varsigma'\neq\varsigma}}^{\nu} a_{\varsigma,\varsigma'}(\boldsymbol{t})x_{\varsigma'}=y_\varsigma+\boldsymbol{t}c(\boldsymbol{t})-2\boldsymbol{t}\partial_{y_\varsigma}\phi+\boldsymbol{t}\sum_{\varsigma'=1}^{\nu}b_{\varsigma,\varsigma'}(\boldsymbol{t})y_{\varsigma'}.$$

Let us consider the above equations as a system of ν equations where the unknowns are $x_1,\dots,x_\nu$. Then there exists $T_2\in]0,T[$ such that, for $\boldsymbol{t}\in]-T_2,T_2[$,

$$x_j=u(\boldsymbol{t})\cdot\partial_y\phi+v(\boldsymbol{t})\cdot y+w(\boldsymbol{t})$$

where u,v (respectively w) are smooth $\mathbb{R}^\nu$-valued (respectively $\mathbb{R}$-valued) functions defined on $]-T_2,T_2[$. These functions and T_2 only depend on the coefficients of the polynomial $P_{\boldsymbol{t}}$. Then, by integration by parts,

$$x_j\mathcal{F}f:=\mathcal{F}\tilde{f} \tag{6.3}$$

where

$$\tilde{f}:=iu(\boldsymbol{t})\cdot\partial_y f+\big(v(\boldsymbol{t})\cdot y+w(\boldsymbol{t})\big)f\in S^{m,k+1}_{r-1}.$$

Moreover there exists $c_3>0$ such that

$$|\tilde{f}|_{m,k+1,p,q,r-1}\leqslant c_3|f|_{m,k,p,q,r}.$$

- Let $k\in\mathbb{R}$ and $r\in\mathbb{N}$. Let $\varphi\in\mathcal{S}(\mathbb{R}^\nu)$ and $u\in\mathcal{C}^\infty_{b,1}(]-T,T[\times\mathbb{R}^{2\nu})$. Let us denote

$$\|\varphi\|_{k,r}:=\sup_{|\gamma|\leqslant r,y\in\mathbb{R}^\nu}(1+|y|)^{-k}|\partial_y^\gamma\varphi|.$$

By Leibniz formula, the function

$$f:(\boldsymbol{t},x,y)\longmapsto u(\boldsymbol{t},x,y)\varphi(y)$$

belongs to $S^{0,k}_r$ and for every $p,q\in\mathbb{N}$, there exists $C>0$ such that

$$|f|_{0,k,p,q,r}\leqslant C\|\varphi\|_{k,r}$$

(C depends on the function u and the numbers k,p,q,r). Let $\boldsymbol{\psi}$ be defined by (3.7). Then $\boldsymbol{\psi}=\mathcal{F}f$. Let $(\alpha,\beta,\delta)\in\mathbb{N}\times\mathbb{N}^\nu\times\mathbb{N}^\nu$. Let us assume that

$$\begin{cases} k+2\alpha+|\beta|+|\delta|<-\nu-1\\ p-\alpha\geqslant 0\ ,\ \ q-|\beta|\geqslant 0\ ,\ \ r-2\alpha-|\beta|-|\delta|\geqslant 0\end{cases}. \tag{6.4}$$

Then, by (6.1), (6.2) and (6.3),

$$x^\delta\partial_{\boldsymbol{t}}^\alpha\partial_x^\beta\psi=\mathcal{F}\tilde{f}$$

where

$$\tilde{f}\in S^{2\alpha+|\beta|,k+2\alpha+|\beta|+|\delta|}_{r-2\alpha-|\beta|-|\delta|}$$

and
$$|\tilde{f}|_{2\alpha+|\beta|,k+2\alpha+|\beta|+|\delta|,p-\alpha,q-|\beta|,r-2\alpha-|\beta|-|\delta|} \leqslant c_1^{\alpha} c_2^{|\beta|} c_3^{|\delta|} C \|\varphi\|_{k,r}.$$
Hence, for $\boldsymbol{t} \in]-T_2, T_2[-\{0\}$ and $x \in \mathbb{R}^{\nu}$,

$$\begin{aligned} \left|x^{\delta}\partial_{\boldsymbol{t}}^{\alpha}\partial_x^{\beta}\boldsymbol{\psi}\right| \leqslant & \quad C_1|4\pi\boldsymbol{t}|^{-\nu/2}(1+|x|)^{2\alpha+|\beta|}\int_{\mathbb{R}^{\nu}}(1+|y|)^{-\nu-1}dy \times \|\varphi\|_{k,r} \\ \leqslant & \quad C_2|\boldsymbol{t}|^{-\nu/2}(1+|x|)^{2\alpha+|\beta|} \times \|\varphi\|_{k,r}. \end{aligned}$$

- Let $p, q, k' \geqslant 0$. Let us choose $k \in \mathbb{R}$ and $r \in \mathbb{N}$ such that

$$\begin{cases} k+2p+q+k' < -\nu-1 \\ \\ r-2p-q-k' \geqslant 0 \end{cases}.$$

Then, if $\alpha \leqslant p$, $|\beta| \leqslant q$ and $|\delta| \leqslant k'$, (6.4) is satisfied and

$$\forall \boldsymbol{t} \in]-T_2, T_2[-\{0\}, \forall x \in \mathbb{R}^{\nu}, \left|x^{\delta}\partial_{\boldsymbol{t}}^{\alpha}\partial_x^{\beta}\boldsymbol{\psi}\right| \leqslant C_3|\boldsymbol{t}|^{-\nu/2}(1+|x|)^{2p+q}$$

where C_3 is a positive number. Let $\bar{k} \in \mathbb{R}$. Then there exists $C_4 > 0$, such that, for $\boldsymbol{t} \in]-T_2, T_2[-\{0\}$ and $x \in \mathbb{R}^{\nu}$,

$$\alpha \leqslant p, |\beta| \leqslant q \Rightarrow \left|\partial_{\boldsymbol{t}}^{\alpha}\partial_x^{\beta}\boldsymbol{\psi}\right| \leqslant C_3|\boldsymbol{t}|^{-\nu/2}(1+|x|)^{-\bar{k}}.$$

If $\kappa > 0$ and g is a smooth function on $]-T_2, T_2[-\{0\}$ satisfying, for every $N \in \mathbb{N}$, $\max_{n\leqslant N}|\partial_{\boldsymbol{t}}^n g(\boldsymbol{t})| \leqslant C_N|\boldsymbol{t}|^{-\kappa}$, then g is smooth on $]-T_2, T_2[$ and

$$\sup_{\boldsymbol{t}\in]-T_2,T_2[,n\leqslant N}|\partial_{\boldsymbol{t}}^n g(\boldsymbol{t})| \leqslant cC_{N+[\kappa]+2}$$

where c only depends on κ and T_2. Therefore, for every $n \in \mathbb{N}$, $\partial_{\boldsymbol{t}}^n\boldsymbol{\psi}(\boldsymbol{t},\cdot) \in \mathcal{S}(\mathbb{R}^{\nu})$ and for every $(k',q) \in \mathbb{R}\times\mathbb{N}$ there exist $(k,r) \in \mathbb{R}\times\mathbb{N}$ and $C > 0$ such that

$$\sup_{\boldsymbol{t}\in]-T_2,T_2[}\|\partial_{\boldsymbol{t}}^n\boldsymbol{\psi}(\boldsymbol{t},\cdot)\|_{k',q} \leqslant C\|\varphi\|_{k,r}.$$

i.e. the mapping $\varphi \longmapsto \boldsymbol{\psi}$ is continuous.

- Let us consider the assertion on $\boldsymbol{\psi}|_{\boldsymbol{t}=0}$. Let $\gamma \in \mathcal{D}(\mathbb{R}^{\nu})$ be such that $\gamma(z) = 1$ if $|z| \leqslant 1$. Let $x \in \mathbb{R}^{\nu}$. Then $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1(y) = \gamma(y-x)\varphi(y)$ and $\varphi_2(y) = \big(1-\gamma(y-x)\big)\varphi(y)$. Since both functions belong to the Schwartz space, it suffices to check the claim for the corresponding $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$. Since φ_2 vanishes on a neighbourhood of x, one gets $\boldsymbol{\psi}_2(\boldsymbol{t},x) = \mathcal{O}(\boldsymbol{t}^{\infty})$ by integrations by parts. Since the support of the function φ_1 is compact, $\boldsymbol{\psi}_2(\cdot,x)|_{\boldsymbol{t}=0} = \varphi_2(x)$. This proves $\boldsymbol{\psi}|_{\boldsymbol{t}=0} = \varphi$.

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Département de Mathématiques, Laboratoire AGM (CNRS), Université de Cergy-Pontoise, 95000 Cergy-Pontoise, France.